

Quantum General Relativity and Hawking Radiation

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Abstract

In a previous paper we have set up the Wheeler-DeWitt equation which describes the quantum general relativistic collapse of a spherical dust cloud. In the present paper we specialize this equation to the case of matter perturbations around a black hole, and show that in the WKB approximation, the wave-functional describes an eternal black hole in equilibrium with a thermal bath at Hawking temperature.

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1 Introduction

Quantum gravitational effects are expected to modify the nature of singularities that arise as the end state of the classical gravitational collapse of a compact object. A concrete analytical model of classical spherical collapse is the Lemaître-Tolman-Bondi (LTB) dust solution of Einstein equations, which shows that the singularity forming in the collapse could be either naked or covered, depending on the choice of initial conditions [1, 2].

Treating this dust collapse model as a classical background, one can quantize a massless scalar field on this space-time using standard techniques [3, 4, 5]. When the classical collapse ends in a black hole, the quantization of the scalar field yields the emission of Hawking radiation from the black hole, as expected. However, a strikingly different result is obtained when the scalar field is quantized on a classical background which ends in a naked singularity. It turns out that during the period of validity of the semi-classical approximation (curvatures should be less than Planck scale), the collapsing cloud emits only about one Planck unit of energy [6]. Moreover, because the back-reaction does not become important so long as gravity can be treated classically, it follows that the future evolution of the star is governed by quantum gravitational effects and it is impossible to say, from the semi-classical approximation, whether the star radiates away its energy on a short time scale or settles down into a black hole state. This is completely different from the black hole case where essentially the entire star evaporates via Hawking radiation during the semi-classical phase.

A full understanding of the gravitational collapse, both in the naked and in the covered case, requires the application of quantum gravity. Given our limited understanding of quantum gravity at present, perhaps the theory which is currently most suited for addressing dynamical quantum collapse and questions regarding its final state is canonical quantum general relativity. Although limited in its ultimate scope as a theory of quantum gravity, the canonical theory *can* meaningfully address the issue of singularities in minisuperspace and midisuperspace models, so long as one can tackle questions relating to operator ordering and regularization, and provide a notion of time evolution in the theory.

The minisuperspace quantization of a collapsing null dust *shell* was analyzed in [7] where it was shown how the classical singularity can be avoided in the quantum theory. In this model, the avoidance is a direct consequence

of the unitary time evolution – since the wave-function vanishes at $r = 0$ for early times, it does so at any time. As a consequence, an ingoing quantum shell develops into a superposition of ingoing *and* outgoing shell. Such a scenario has interesting physical features that cannot be seen in a semi-classical approximation: no event horizon can form, and there is neither an information loss nor a naked singularity.

A midisuperspace program to study the canonical quantum dynamics of the LTB dust collapse has been developed during the last two years [8], following earlier pioneering work by Kuchař [9] on the quantization of the Schwarzschild geometry. Here one sets up a canonical description of the collapse, using the dust proper time, the area radius and the mass function of the cloud as canonical configuration variables. The evolution is recorded by the dust proper time. One then develops the quantization via the momentum constraint and the Wheeler-DeWitt equation, to which a solution for a general mass function has been found using an ad hoc delta-function regularization used by DeWitt. We show below that this regularization scheme is equivalent to the WKB approximation. (A similar analysis has also been carried out for null dust [10] although the constraints that are obtained in this case are linear and there is no need for regularization.)

The Schwarzschild black hole can be viewed as a LTB model with a constant mass function. We have applied [11] the program described above to this particular case and shown how the horizon area quantization and entropy of the eternal black hole can be understood in terms of quantized shells of matter that are trapped inside the horizon.

The purpose of the present paper is to show that the WKB solution is able also to describe an eternal black hole in equilibrium with a thermal bath at Hawking temperature, if the mass function is chosen appropriately. We argue that Hawking radiation may be understood as a combination of the WKB and the Born-Oppenheimer approximations on the full quantum wave-functional. It reinforces our belief in the overall consistency of the program and, in particular, suggests that our proposed choice of operator ordering and our definition of the inner product on the Hilbert space of wave-functionals capture features of the full theory. As we will see below, the definition of the inner product enters crucially in the calculation of the Hawking radiation.

In Section 2, we briefly recall key results from our previous paper on quantum dust collapse [8], leading up to the WKB solution of the Wheeler-deWitt equation. In Section 3, this solution is specialized to the case of

matter around a black hole, and shown to describe Hawking radiation.

2 Canonical quantization of dust collapse

The spherical gravitational collapse of a dust cloud having energy density $\epsilon(\tau, \rho)$ in an asymptotically flat space-time is described in comoving coordinates $(\tau, \rho, \theta, \phi)$ by the LTB metric

$$ds^2 = d\tau^2 - \frac{\tilde{R}^2(\rho, \tau)}{1 + f(\rho)} d\rho^2 - R^2(\rho, \tau) d\Omega^2, \quad (1)$$

and the Einstein equations

$$\epsilon(\tau, \rho) = \frac{\tilde{F}}{R^2 \tilde{R}}, \quad R^*(\tau, \rho) = \pm \sqrt{f + F/R}. \quad (2)$$

Here, $F(\rho)$ is twice the mass to the interior of the coordinate ρ and $R(\tau, \rho)$ is the area radius of the shell labeled ρ at the dust proper time τ . A tilde and an asterisk denote partial derivatives with respect to ρ and τ , respectively. (Throughout this paper, the gravitational constant is set equal to one.)

The canonical dynamics of the collapsing cloud is described by embedding the spherically symmetric ADM 4-metric,

$$ds^2 = N^2 dt^2 - L^2 (dr - N^r dt)^2 - R^2 d\Omega^2, \quad (3)$$

in the LTB space-time (1), and by casting the action for the Einstein-dust system in canonical form. The phase space of non-rotating dust is described by the dust proper time, τ , and its conjugate momentum, whereas the gravitational phase space consists of the configuration space variables (R, L) and their conjugate momenta. Using a version of the canonical transformation developed by Kuchař [9], the configuration variable L is replaced by a new variable F (the mass function). In terms of the new chart $(\tau, R, F, P_\tau, P_R, P_F)$, the momentum and the Hamiltonian constraints read [8]

$$H_r = \tau' P_\tau + R' P_R + F' P_F \approx 0, \quad (4)$$

$$H = P_\tau^2 + \mathcal{F} P_R^2 - \frac{F'^2}{4\mathcal{F}} \approx 0. \quad (5)$$

Here, $\mathcal{F} \equiv 1 - F/R$. The Hamiltonian constraint shows that on the effective configuration space (τ, R) , the DeWitt super-metric is just $\text{diag}(1, 1/\mathcal{F})$. This is a flat metric, therefore a redefinition of the area coordinate according to

$$R_* = \pm \int \frac{dR}{\sqrt{|\mathcal{F}|}} , \quad (6)$$

where the positive sign refers to the region exterior to the horizon ($R > F$) and the negative sign to the region interior to the horizon ($R < F$), brings the super-metric to manifestly flat form. In terms of the momentum P_* , conjugate to R_* , the Hamiltonian constraint reads

$$H = P_\tau^2 \pm P_*^2 - \frac{F'^2}{4\mathcal{F}} \approx 0 . \quad (7)$$

Quantization is implemented by raising the momenta to operator status and requiring the physical state, $\Psi[\tau, R_*, F]$, to be annihilated by the constraints. In this way, the time evolution of $\Psi[\tau, R_*, F]$ is determined by the Hamiltonian constraint,

$$\left[\frac{\delta^2}{\delta\tau^2} \pm \frac{\delta^2}{\delta R_*^2} + \frac{F'^2}{4\mathcal{F}} \right] \Psi(\tau, R, F) = 0 \quad (8)$$

(where the positive sign before the second term refers to the region outside the horizon, and the negative sign to the region inside), while invariance under spatial diffeomorphisms is implemented by the momentum constraint,

$$\left[\tau' \frac{\delta}{\delta\tau} + R'_* \frac{\delta}{\delta R_*} + F' \frac{\delta}{\delta F} \right] \Psi[\tau, R, F] = 0 . \quad (9)$$

In the region exterior to the horizon, Eq. (8) is no longer hyperbolic, in contrast to the Wheeler-DeWitt equation on the original configuration space. The reason lies in the canonical transformations performed, which lead to a new effective configuration space.

To complete the quantum theory, one must define an inner product on the Hilbert space of wave-functionals. In [8], we defined it in a natural way by exploiting the fact that the DeWitt super-metric is manifestly flat in the configuration space (τ, R_*) ,

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{R_*(0)}^{\infty} dR_* \bar{\Psi}_1 \Psi_2 . \quad (10)$$

Note that this inner product is defined on a $\tau = \text{constant}$ hypersurface. Equations (8)–(10) clearly imply a specific choice, albeit a natural one, of operator ordering. We will see below that this choice is sufficient to reproduce the Hawking effect.

The momentum constraint is obeyed by *any* functional that is a spatial scalar and, in particular, by the functional

$$\Psi[\tau, R, F] = \exp \left[\frac{1}{2} \int_0^\infty dr F'(r) \mathcal{W}(\tau(r), R(r), F(r)) \right] , \quad (11)$$

provided that \mathcal{W} has no explicit dependence on the radial label coordinate, r . Our choice, while not unique, is dictated by the knowledge that F' is the proper energy density of the collapsing cloud. When (11) is substituted in the Wheeler-DeWitt equation (8) one finds, on using DeWitt's δ -function regularization ($\delta(0) = 0 = \delta^{(n)}(0) \forall n \in \mathbf{N}$), that \mathcal{W} obeys

$$\frac{F'^2}{4} \left[\left(\frac{\partial \mathcal{W}}{\partial \tau} \right)^2 \pm \left(\frac{\partial \mathcal{W}}{\partial R_*} \right)^2 + \frac{1}{\mathcal{F}} \right] \Psi = 0 . \quad (12)$$

We emphasize again that this regularization prescription is at this stage completely ad hoc, and could only be justified from an understanding of the full theory. It is even imaginable that, analogous to string theory, Schwinger terms may arise in the commutation relations of the constraints that could forbid the implementation of the Wheeler-DeWitt equation [12]. Fortunately, however, for our present purpose of recovering Hawking radiation it is not necessary to resolve this issue.

After defining $\mathcal{W} = -i\tau + \mathcal{U}(R, F)$, (12) yields the solution

$$\mathcal{W} = -i\tau \pm 2i\sqrt{F} \left[\sqrt{R} - \sqrt{F} \tanh^{-1} \sqrt{\frac{F}{R}} \right] \quad (13)$$

outside the horizon, and

$$\mathcal{W} = -i\tau \mp 2i\sqrt{F} \left[\sqrt{R} - \sqrt{F} \tanh^{-1} \sqrt{\frac{R}{F}} \right] \quad (14)$$

inside.

3 Origin of Hawking radiation

We begin by noting that the quantum constraint (12), which has been written using DeWitt's delta-function regularization, is the same equation as one would get by writing the Wheeler-DeWitt equation (8) in the highest-order WKB approximation. The reason is that this prescription effectively suppresses the (divergent) WKB prefactor. To show this, let us expand the wave-functional Ψ of (8) (with \hbar being re-inserted) in a power-series in \hbar ,

$$\Psi \equiv \exp(iS/\hbar) \equiv \exp \left[\frac{1}{2\hbar} \int_0^\infty dr F'(r) \mathcal{S}(\tau(r), R(r), F(r)) \right], \quad (15)$$

$$\mathcal{S}(\tau, R, F) = \mathcal{S}_0 + \hbar \mathcal{S}_1 + \hbar^2 \mathcal{S}_2 + \dots \quad (16)$$

Substituting this expansion in (8) and retaining only the leading order, \hbar -independent, terms gives

$$\frac{F'^2}{4} \left[\left(\frac{\partial \mathcal{S}_0}{\partial \tau} \right)^2 \pm \left(\frac{\partial \mathcal{S}_0}{\partial R_*} \right)^2 - \frac{1}{\mathcal{F}} \right] \Psi = 0. \quad (17)$$

Comparison with (12) shows that the WKB solution is the same as one would get by doing the delta-function regularization in the original Wheeler-DeWitt equation (after the identification $\mathcal{W} = i\mathcal{S}_0$). (For a similar discussion of WKB states for the Schwarzschild black hole see [13] and for two-dimensional dilaton gravity see [14]).

We will now show that the wave-functional (11), along with the solutions (13) and (14), yields Hawking radiation when it is applied to a matter distribution that is appropriate to a massive black hole surrounded by dust whose total energy is small compared with the mass of the black hole. For this purpose, let us assume that the mass function $F(r)$ is of the form

$$F(r) = 2M\theta(r) + f(r), \quad (18)$$

where $\theta(r)$ is the Heaviside step-function, and $f(r)$ (not to be confused with $f(\rho)$ occurring in (1)) is differentiable, representing a dust distribution with $f(r)/2M \ll 1$. This mass function is interpreted as the presence of a Schwarzschild black hole of mass M at the origin, and $f(r)$ is a dust matter perturbation on the black hole, which, as we now show, can be related to

Hawking radiation. In a sense, it plays the role of the quantum field used in standard derivations of Hawking radiation.

Inserting this mass function in the wave-functional (11) gives (setting $\hbar = 1$ again)

$$\Psi[\tau, R, F] = \exp \left[\frac{1}{2} \int_0^\infty dr [2M\delta(r) + f'(r)] \mathcal{W}(\tau(r), R(r), F(r)) \right] \quad (19)$$

or

$$\Psi[\tau, R, F] \equiv \exp [M\mathcal{W}_0(\tau, R, F)] \times \exp \left[\frac{1}{2} \int_0^\infty dr f'(r) \mathcal{W}^f(\tau(r), R(r), F(r)) \right] \quad (20)$$

where $\mathcal{W}_0(\tau, R, F) \equiv \mathcal{W}(\tau(0), R(0), F(0))$. The first exponent on the right-hand side is the WKB wave-functional representing the black hole at the origin, as shown in [11]. The second term, up to order $f(r)$, represents a matter distribution that propagates in this background, if we take $F(r) \approx 2M$ in \mathcal{W}^f . Thus we have

$$\Psi = \Psi_{\text{bh}} \times \Psi_f, \quad (21)$$

where

$$\begin{aligned} \Psi_{\text{bh}} &\sim \exp [M\mathcal{W}_0(\tau, R, F)] , \\ \Psi_f &\sim \exp \left[\frac{1}{2} \int_0^\infty dr f'(r) \mathcal{W}^f(\tau(r), R(r), F(r)) \right] . \end{aligned}$$

The solution Ψ_f is known from (11), (13) and (14) above and given by

$$\mathcal{W}_{\text{out}}^f = -i \left[\tau \mp 2\sqrt{2M} \left(\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left[\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{R} - \sqrt{2M}} \right] \right) \right] \quad (22)$$

outside the horizon, and

$$\mathcal{W}_{\text{in}}^f = -i \left[\tau \pm 2\sqrt{2M} \left(\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left[\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{2M} - \sqrt{R}} \right] \right) \right] \quad (23)$$

inside the horizon.

We would like to rewrite the expressions for \mathcal{W}^f in terms of the Killing time T . For the Schwarzschild background being considered for the distribution $f(r)$ and for contracting clouds, we have the following relation between

the proper time and the Killing time (see e.g. [15]),

$$\tau = T + \sqrt{2M} \int dR \frac{\sqrt{R}}{R - 2M} \quad (24)$$

$$= T + 2\sqrt{2M} \left[\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left(\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{R} - \sqrt{2M}} \right) \right] . \quad (25)$$

Thus, in terms of T , there are two possibilities for $\mathcal{W}_{\text{out}}^f$,

$$\mathcal{W}_{\text{out}}^f = \begin{cases} -iT , \\ -i \left[T + 4\sqrt{2M} \left(\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left[\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{R} - \sqrt{2M}} \right] \right) \right] . \end{cases} \quad (26)$$

The wave-functional that corresponds to an infalling wave at $T \rightarrow -\infty$ and $R \rightarrow \infty$ is the one with $\mathcal{W}_{\text{out}}^f$ given by the second of the above. We will therefore concentrate on

$$\Psi_f = \exp \left[-i \int dr f'(r) \left(T + 4\sqrt{2M} \left(\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left[\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{R} - \sqrt{2M}} \right] \right) \right) \right] . \quad (27)$$

Defining $Z = 4\sqrt{2MR}$ we find that as $R \rightarrow \infty$ this wave-functional approaches

$$\Psi_f^- \approx \exp \left[-i \int dr f'(r) [T + Z] \right] \quad (28)$$

which undergoes rapid oscillations except when $T \rightarrow -\infty$, that is on \mathcal{I}^- . When $T \rightarrow \infty$, in order for the phase to be not large we see that $R \rightarrow 2M$, i.e. $Z \rightarrow 8M$. In this limit we have

$$\begin{aligned} \Psi_f^+ &= \exp \left\{ -i \int dr f'(r) \left(T + \left[Z - 4M \ln \left(\frac{Z + 8M}{Z - 8M} \right) \right] \right) \right\} \\ &\approx \exp \left\{ -i \int dr f'(r) \left(T + 4M \ln \left(\frac{Z - 8M}{16M} \right) \right) \right\} . \end{aligned} \quad (29)$$

This is similar to what happens in the geometric optics approximation. The simple looking phase on \mathcal{I}^- has scattered through the geometry to turn into the complicated looking phase on \mathcal{I}^+ near the horizon.

Equation (28) represents infalling waves. We can think of it as a product over plane waves, one at each label r , as follows:

$$\Psi_\omega^- = \prod_r e^{-i\omega(r)[T(r)+Z(r)]} . \quad (30)$$

This should represent a complete set of infalling modes at each label r , if we think of the $\omega(r)$ as the frequency of the modes. In other words, we allow all possible $\omega(r) = \Delta f(r)$.

A complete set of outgoing modes on \mathcal{I}^+ would likewise be given by the functional

$$\Psi_{\omega}^+ = \prod_r e^{-i\omega(r)[T(r)-Z(r)]} . \quad (31)$$

This is because the transformation from the dust proper time to the Killing time is now obtained by matching an expanding (rather than contracting) dust cloud to a Schwarzschild exterior, for which one gets, instead of (25), the relation

$$\tau = T - \sqrt{2M} \int dR \frac{\sqrt{R}}{R-2M} \quad (32)$$

$$= T - 2\sqrt{2M} \left[\sqrt{R} - \frac{\sqrt{2M}}{2} \ln \left(\frac{\sqrt{R} + \sqrt{2M}}{\sqrt{R} - \sqrt{2M}} \right) \right] . \quad (33)$$

It is then easily shown that as $R \rightarrow \infty$, the asymptotic form of the wave-functional is as in (31).

Now we ask the question: what is the projection of our solution (29) on the negative frequency modes of the outgoing basis on \mathcal{I}^+ . For this purpose we must consider the inner product of states on a hypersurface of constant Schwarzschild time T . Thus we must transform from the Euclidean flat metric on the (R_*, τ) plane of Eqn. (8) to the metric in the (R, T) coordinates. Using the relations (6) and (25) we get

$$g_{RR} = \left(\frac{\partial R_*}{\partial R} \right)^2 g_{R_*R_*} + \left(\frac{\partial \tau}{\partial R} \right)^2 g_{\tau\tau} = \left(\frac{R}{R-2M} \right)^2$$

(Note that in (8) the positive sign holds outside the horizon, so that $g_{R_*R_*} = +1$.) The required inner product on a constant T hypersurface then is

$$\begin{aligned} \langle \Psi_{\omega} | \Psi_f^+ \rangle &= \prod_r \int \sqrt{g_{RR}} dR(r) \Psi_{\omega}^+(Z(r), T(r), \omega(r)) \Psi_f^+(Z(r), T(r), f(r)) \\ &= \prod_r \int dR \frac{R}{R-2M} \Psi_{\omega}^+(Z(r), T(r), \omega(r)) \Psi_f^+(Z(r), T(r), f(r)) \\ &= \prod_r \int dZ \frac{Z^3}{16M(Z^2 - 64M^2)} \Psi_{\omega}^+(Z, T, \omega) \Psi_f^+(Z, T, f) \end{aligned}$$

$$\approx \prod_r \int dZ \frac{2M}{Z-8M} \Psi_\omega^+(Z, T, \omega) \Psi_f^+(Z, T, f) . \quad (34)$$

This projection represents the negative frequency modes present in the solution. We are interested in $|\langle \Psi_\omega^+ | \Psi_f^+ \rangle|^2$ because these are the analogs of the Bogoliubov coefficients, $|\beta(f, \omega)|^2$. If we think of $\beta(f, \omega)$ as

$$\beta(f, \omega) = \prod_r \beta(f(r), \omega(r)) , \quad (35)$$

we have

$$\begin{aligned} \beta(f(r), \omega(r)) &\approx 2M \int_{8M}^\infty \frac{dZ}{Z-8M} e^{i\omega Z} e^{-4iM\Delta f \ln(\frac{Z-8M}{16M})} \\ &= \frac{2M}{(16M)^{-4iM\Delta f}} \int_{8M}^\infty dZ (Z-8M)^{-1-4iM\Delta f} e^{i\omega Z} . \end{aligned} \quad (36)$$

Substituting $u = Z - 8M$ and integrating we find

$$\beta(f(r), \omega(r)) = \frac{2M e^{8M i \omega}}{(16M)^{-4iM\Delta f}} \left[\Gamma(-i\sigma) \omega^{i\sigma} e^{-\sigma\pi/2} \right] ,$$

where $\sigma = 4M\Delta f$. This gives

$$|\beta(f, \omega)|^2 = \prod_r \frac{2\pi M}{\Delta f} \left[\frac{1}{e^{8\pi M\Delta f} - 1} \right] . \quad (37)$$

This is interpreted as the eternal black hole being in equilibrium with a thermal bath at the Hawking temperature $(8\pi M)^{-1}$. Our derivation provides a functional Schrödinger picture for dust Hawking radiation, consistent with the WKB wave-functional which solves the Wheeler-DeWitt equation.

4 Concluding remarks

In this paper we have obtained a derivation of Hawking radiation for dust matter, starting from the WKB wave-functional which satisfies the Wheeler-DeWitt equation for quantum spherical dust collapse. The fact that such a derivation could be found should be treated as support for the validity of the inner product defined in Equation (10).

A functional description of Hawking radiation can also be given within a Born-Oppenheimer type of approximation to quantum gravity. Instead of our wave function Ψ_f , one has there a Gaussian quantum state for a quantum field. Evolving this state through the background of an object collapsing to a black hole, one finds that it encodes information about the Hawking radiation similar to our Ψ_f [16].

The present analysis also suggests that an exact treatment of the Wheeler-DeWitt equation (8) which goes beyond the WKB approximation should yield corrections to Hawking radiation, and provide a better understanding of the end state of gravitational collapse. This could perhaps be done along the general lines presented in [17] in which corrections to the semiclassical limit have been calculated from the Wheeler-DeWitt equation. These issues are at present under investigation.

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References

- [1] T. P. Singh and P. S. Joshi, *Class. Quantum Grav.* 13, 559 (1996).
- [2] S. Barve, T. P. Singh, C. Vaz, and L. Witten, *Class. Quantum Grav.* 16, 1727 (1999).
- [3] S. Barve, T. P. Singh, C. Vaz, and L. Witten, *Nucl. Phys.* B532, 361 (1998).
- [4] S. Barve, T. P. Singh, C. Vaz, and L. Witten, *Phys. Rev.* D58, 104018 (1998).
- [5] S. Barve, T. P. Singh, and C. Vaz, *Phys. Rev.* D62, 084021 (2000).

- [6] T. Harada, H. Iguchi, K. Nakao, T. P. Singh, T. Tanaka, and C. Vaz, Phys. Rev. D64, 041501 (2001).
- [7] P. Hájíček and C. Kiefer, Nucl. Phys. B603, 531 (2001); P. Hájíček, *ibid.*, 555 (2001); P. Hájíček and C. Kiefer, Int. J. Mod. Phys. D10, 775 (2001); P. Hájíček, [gr-qc/0204049](#).
- [8] C. Vaz, L. Witten, and T. P. Singh, Phys. Rev. D63, 104020 (2001).
- [9] K. V. Kuchař, Phys. Rev. D50, 3961 (1994); see also H. A. Kastrup and T. Thiemann, Nucl. Phys. B425, 665 (1994).
- [10] C. Vaz, L. Witten, and T. P. Singh, Phys. Rev. D65, 104016 (2002).
- [11] C. Vaz and L. Witten, Phys. Rev. D64, 084005 (2001).
- [12] D. Cangemi, R. Jackiw, and B. Zwiebach, Ann. Phys. (N.Y.) 245, 408 (1996).
- [13] T. Brotz and C. Kiefer, Phys. Rev. D55, 2186 (1997).
- [14] D. Louis-Martinez, J. Gegenberg, and G. Kunstatter, Phys. Lett. B321, 193 (1994).
- [15] L. H. Ford and L. Parker, Phys. Rev. D17, 1485 (1978).
- [16] J.-G. Demers and C. Kiefer, Phys. Rev. D53, 7050 (1996).
- [17] C. Kiefer and T. P. Singh, Phys. Rev. D44, 1067 (1991); A. O. Barvinsky and C. Kiefer, Nucl. Phys. B526, 509 (1998).